ECE 346: Digital Signal Processing
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Week of March 29, 2021

Recitation #8: Linear Phase, Real-Valued FIR Filters

Objective & Outline

- Problems $1 - 4$: recitation problems
- Problem 5: self-assessment problem

Let's first go over some notes on linear phase, real-valued FIR filters:

- Given frequency response $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$, a filter has linear phase if its phase is defined as
$$\angle H(e^{j\omega}) = -\alpha\omega \pm \beta, \tag{1}$$
where α and β are constants. This is because linear phase is often described as by a constant group delay, where the group delay is defined as
$$\tau(\omega) = -\frac{d\angle H(e^{j\omega})}{d\omega}. \tag{2}$$
- If a filter is a linear-phase, causal FIR filter, then if $z = r$ is a zero, then
$$z = (re^{j\phi})^{-1} = \frac{1}{r}e^{-j\phi} \tag{3}$$
must also be a zero.
- Note that a filter can have zeroes on the unit circle (e.g. $z = -1, 1$) or at zero and still be a filter with linear phase, as the inverse of a zero on the unit circle is itself.
- If a filter is a real-valued FIR filter, then if $z = re^{j\phi}$ is a zero, then its complex conjugate
$$z = re^{-j\phi} \tag{4}$$
must also be a zero.
- Putting these facts together, if a filter is a real-valued, linear phase FIR filter and a zero is NOT real and NOT on the unit circle, then the zero must come in multiples of 4.
- An even length real-valued, linear phase FIR filter must have a zero at $z = 1$ and/or at $z = -1$.
- We conclude these notes by talking about the **four types of real-valued, linear-phase FIR filters:**
  1. Type 1: Odd-length filter with a symmetric impulse response (used for either LPF or HPF)
  2. Type 2: Even-length filter with a symmetric impulse response (not good for HPF)
  3. Type 3: Odd-length filter with a anti-symmetric impulse response (not good for either LPF or HPF)
  4. Type 4: Even-length filter with a anti-symmetric impulse response (not good for LPF)

The problems start on the following page.

Problem 1 (Review)**.** These problems should be a review of lecture videos and notes.

1. Which of the following can be linear phase filters?
   (a) $h_1[n] = \delta[n-10]$
   (b) $h_2[n] = \delta[n] - \delta[n-1] + 2\delta[n-2]$
   (c) $h_3[n] = \delta[n-4] + \delta[n-5] + \delta[n-6]$
2. Which of the following cannot be high-pass filters?
   (a) $h_1[n] = u[n] - u[n-10]$
   (b) $h_2[n] = (n-6)(u[n] - u[n-13])$
   (c) $h_3[n] = \delta[n] - \delta[n-1]$
   (d) $h_4[n] = u[n] - u[n-11]$

Problem 1:

① There are three characteristics to look for in there impulse responses to see if the filter has linear phase:

1) constant group delay
2) the inverse of a zero is also a zero of the filter
3) check the symmetry (or anti-symmetry) of the impulse response.

With the first two satisfied, we just need to check the symmetry of the impulse responses:

(a) $h_1[n] = \{\ldots, 0, \underset{\nearrow \atop n=10}{1}, 0, \ldots\}$

By looking at the sequence, $h_1[n]$ is symmetric and is linear phase.

(b) $h_2[n] = \{1, -1, 2\}$

$h_2[n]$ is neither symmetric nor anti-symmetric, and thus is not a linear phase filter.

(c) $h_3[n] = \{1, 1, 1\}$.

By looking at the sequence, $h_3[n]$ is symmetric and is linear phase.

② Since Type 2, Type 3 filters are not HPF, we just need to check the types of the filters:

(a) $h_1[n]$ is length - 10 (even) $\}$ Type 2 filter ⇒ cannot be HPF.
" is symmetric

(b) $h_2[n]$ is length - 13 (odd) $\}$ Type 3 filter ⇒ cannot be HPF.
is anti-symmetric

(c) $h_3[n]$ is length - 2 (even) $\}$ Type 4 filter ⇒ can be HPF.
is anti-symmetric

(d) $h_4[n]$ is length - 11 (odd) $\}$ Type 1 filter ⇒ can be HPF.
is symmetric

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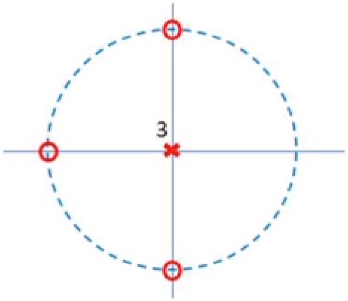
Problem 2 (FIR Filters)**.** Answer the following questions given the pole-zero diagram of a digital filter shown below:



(a) Is this an IIR or FIR filter?

(b) Is the filter causal or non-causal?

(c) What is the region of convergence for this filter?

(d) What kind of filter is this system (e.g. low-pass, high-pass)?

(e) Provide the transfer function of the filter under the assumption that $|H(e^{j0})| = 1$.

Problem 2:

(a) Since the filter only has trivial poles (i.e. poles at 0 or ∞), the filter is an FIR filter.

(b) Since the filter has all poles at and are inside the unit circle, the filter is causal.

(c) As the filter has all poles at $z=0$ and is causal, the ROC is

$$|z| > 0.$$

(d) The filter is a low-pass filter, as there is attenuation near and at $\omega = \pi$ and no attenuation at $\omega = 0$. We can see this by the location of the zeroes.

(e) $$H(z) = \underbrace{p_0}_{\text{gain factor}} \cdot [\text{ the zeros of the filter }]$$

$$= p_0 (1 - jz^{-1})(1 + jz^{-1})(1 + z^{-1})$$

$$= p_0 (1 + z^{-1} + z^{-2} + z^{-3})$$

Using the fact that $|H(e^{j0})| = 1$, we can solve for the gain factor:

$$|H(e^{j0})| = |H(1)| = p_0 [1 + 1 + 1 + 1] = 1$$

$$= 4p_0 = 1$$

$$\Rightarrow p_0 = 1/4.$$

And thus,

$$H(z) = \tfrac{1}{4}(1 + z^{-1} + z^{-2} + z^{-3}).$$

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Problem 3 (FIR Filters)**.** Suppose that a length-5 causal FIR filter had two second order zeroes at

$$z = 0.95 \tag{5}$$
$$z = 1.05. \tag{6}$$

(a) Provide a pole-zero diagram for this filter.

(b) What kind of filter is this system (e.g. low-pass, high-pass)?

(c) Provide the transfer function $H(z)$ for this filter.

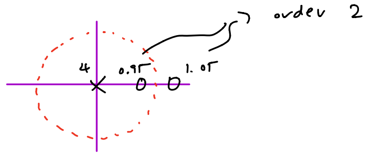
(d) Assuming the gain $p_0 = 2$, what is $|H(e^{j0})|$?

(e) Provide a sketch of the magnitude response of the filter.

(a) Since the filter is causal, we have a pole of order 4 at $z = 0$, as

$$\# \text{ of zeroes} = \# \text{ of poles}.$$

The pole-zero diagram is given by



(b) This filter is a high-pass filter, as the zeroes cause attenuation at $\omega = 0$.

(c) Similar to the previous problem:

$$H(z) = p_0 (1 - 0.95 z^{-1})^2 (1 - 1.05 z^{-1})^2.$$

(d) Now given $p_0 = 2$:

$$H(z) = 2 (1 - 0.95 z^{-1})^2 (1 - 1.05 z^{-1})^2.$$

Using the fact that $|H(e^{j0})| = |H(1)|$, we can plug in $z^{-1} = 1$:

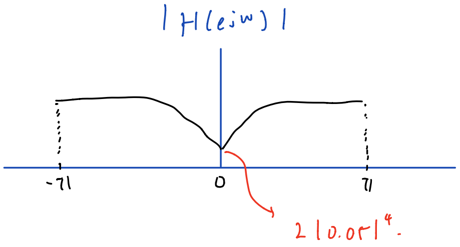
$$|H(1)| = 2 \, | 1 - 0.95 (1) |^2 \cdot | 1 - 1.05 (1) |^2$$

$$= 2 \, | 0.05 |^2 \, | 0.05 |^2$$

$$= 2 \cdot | 0.05 |^4.$$

Thus, $|H(e^{j0})| = 2 \cdot |0.05|^4$.

(e)



Problem 4 (FIR Filters)**.** Assuming minimum filter length, suppose that a causal, real-valued, linear-phase FIR filter had known zeroes at

$$z = 1 \tag{7}$$

$$z = 2e^{j\frac{\pi}{3}}. \tag{8}$$

(a) Provide a pole-zero diagram for this filter.

(b) If the gain of this filter was $p_0 = 2$, provide the impulse response of the filter.

(c) What *type* of filter is this FIR filter (e.g. Type I, Type II, etc.)?

Problem 4:

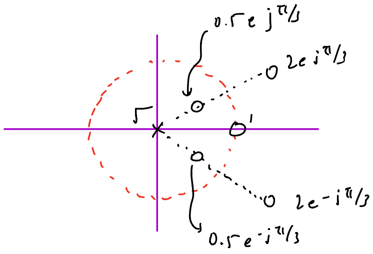
(a) We are given the zeroes

$$z = 1$$
$$z = 2e^{j\pi/3}.$$

Since $z = 2e^{j\pi/3}$ is NOT on the unit circle and NOT real-valued, the zero must come in multiples of 4, as we have a real-valued, linear phase filter. Thus we have zeroes

$$z = 2e^{j\pi/3} \qquad z = 0.5e^{j\pi/3}$$
$$z = 2e^{-j\pi/3} \qquad z = 0.5e^{-j\pi/3}.$$

The pole - zero diagram is



(b) We have

$$H(z) = 2(1-z^{-1})(1-2e^{j\pi/3}z^{-1})(1-2e^{-j\pi/3}z^{-1})(1-0.5e^{j\pi/3}z^{-1})(1-0.5e^{-j\pi/3}z^{-1})$$
$$= 2 - 2z^{-1} + 8.5z^{-2} - 8.5z^{-3} + 2z^{-4} - 2z^{-5}.$$

Taking the inverse z-transform is reading off the terms:

$$h[n] = \{2, -2, 8.5, -8.5, 2, -2\}.$$

(c) The impulse response

$$h[n] = \{2, -2, 8.5, -8.5, 2, -2\}$$

is even length and anti-symmetric, and is a

Type 4 filter.

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Problem 5 (Self-assessment)**.** As usual, try to work on these problems together in break-out rooms.

1. Consider an FIR filter with real-valued coefficients whose z-transform is given by

$$H(z) = a + bz^{-2}. \tag{9}$$

   The impulse response of this filter has unit energy with $a \geq 0$ and

$$H(e^{j0}) = H(e^{j\pi}) = 0. \tag{10}$$

   (a) Provide a pole-zero diagram for this filter.

   (b) What *type* of filter is this FIR filter (e.g. Type I, Type II, etc.)?

   (c) What is the impulse response of this filter?

   (d) Provide a closed-form expression for $H(e^{j\omega})$ and provide its labeled plot.

Problem 5:

Before we begin, we should understand what we're given:

$$H(z) = a + bz^{-2} \leftrightarrow h[n] = \{\underbrace{a,}_{n=0}\ 0,\ b\}$$

$$\text{unit energy} \Rightarrow a^2 + b^2 = 1$$

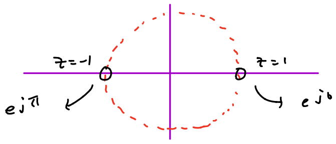
$$H(e^{j0}) = H(e^{j\pi}) = 0 \Rightarrow a + b(\pm 1)^{-2} = 1$$

$$\Rightarrow a = -b.$$

$$a \geq 0.$$

(a) The zeroes are given by the fact

$$H(e^{j0}) = H(e^{j\pi}) = 0 :$$



(b) Recall that the impulse response is

$$h[n] = \{a, 0, b\}$$

and that $a = -b$. Thus $h[n]$ is odd length and anti-symmetric, which is a type 3 filter.

(c) We need to solve for a and b:

$$a = -b$$

$$\Rightarrow (-b)^2 + b^2 = 1$$

$$\Rightarrow b^2 + b^2 = 1$$

$$\Rightarrow 2b^2 = 1 \Rightarrow b = \pm \frac{1}{\sqrt{2}}.$$

Since $a \geq 0$, then $a = \frac{1}{\sqrt{2}}$ and $b = -\frac{1}{\sqrt{2}}$. Thus,

$$h[n] = \{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\}.$$

(d) Using $z = e^{j\omega}$:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} e^{-j2\omega}.$$

$\blacksquare$